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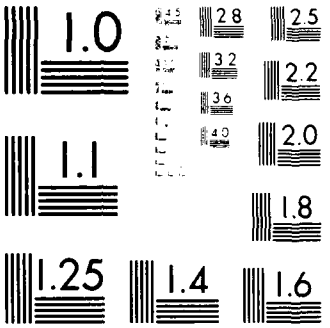

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A NOTE ON EXTENDED QUASI-LIKELIHOOD

by

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A NOTE ON EXTENDED QUASI-LIKELIHOOD

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SUMMARY

We study the method of Extended Quasi-Likelihood estimation of a variance function. This method is shown to be closely related to the method of pseudo-likelihood estimation as in Carroll & Ruppert (1982).

Keywords: EXPONENTIAL FAMILY; HETEROSCEDASTIC REGRESSION MODEL; INFERENCE FOR VARIANCE PARAMETERS; PSEUDO-LIKELIHOOD ESTIMATION; VARIANCE FUNCTION ESTIMATION.

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1. INTRODUCTION

Consider the following mean-variance model for observable data y:

$$E(y_i) = \mu_i = \mu_i(\beta) = f(x_i,\beta) \; ; \; var(y_i) = \sigma^2 g^2(\mu_i, z_i, \theta). \qquad (1.1)$$

Here, y_i is the ith response variable of N independent observations, (x_i, z_i) are associated vectors of covariates, f is the regression function, β is a p-vector of regression parameters, σ is a scale parameter, and g is the variance function with variance parameter θ ($r \times 1$). For example, the variance may be modeled as proportional to an unknown power of the mean:

$$g(\mu_i, z_i, \theta) = \mu_i^{\theta}, \quad \mu_i > 0. \qquad (1.2)$$

Special cases of (1.1) are used in applications such as radioimmunoassay, econometrics, and chemical kinetics. Model (1.1) includes the class of generalized linear models, see McCullagh & Nelder (1983).

A usual aim is the estimation of β, with estimation of the variance function parameters as an adjunct. However, as discussed by Davidian & Carroll (1987) and Carroll, Davidian & Smith (unpublished), estimation of the variance function, in particular the parameter θ, is an important problem both for estimation of β and in its own right.

Most methods for estimating θ are "regression" methods based on generalized least squares. In these techniques, θ and σ are estimated by a weighted regression of some function of the absolute residuals from a fit $\hat{\beta}_*$ on their expectations. For example, in location-scale problems squared residuals have approximate mean proportional to $g^2(\mu_i, z_i, \theta)$ and variance proportional to $g^4(\mu_i, z_i, \theta)$. Thus an estimate of θ can be obtained by a

generalized least squares regression of squared residuals on $\sigma^2 g^2(\hat{\mu}_i, z_i, \theta)$ with variance function $g^4(\hat{\mu}_i, z_i, \theta)$, where $\hat{\mu}_i = f(x_i, \hat{\beta}_*)$. A related method is the pseudo-likelihood approach of Carroll & Ruppert (1982). In this method, one pretends $\beta = \hat{\beta}_*$ and then estimates (σ, θ) by normal theory maximum likelihood, maximizing $\ell_{PL}(\hat{\beta}_*, \theta, \sigma)$, where

$$\ell_{PL}(\beta, \theta, \sigma) = -N \log \sigma - \sum_{i=1}^{N} \log [g\{\mu_i(\beta), z_i, \theta\}] - (2\sigma^2)^{-1} \sum_{i=1}^{N} \{y_i - f(x_i, \beta)\}^2 / g^2\{\mu_i(\beta), z_i, \theta\} . \quad (1.3)$$

This process may be iterated with a generalized least squares routine for β. The pseudo-likelihood method is asymptotically equivalent to weighted regression on squared residuals, and full iteration of such a regression yields the pseudo-likelihood estimate. Both methods can be modified to account for loss of degrees of freedom for preliminary estimation of β as in Harville (1977); for a discussion and a review of many common methods for estimation of θ, see Davidian & Carroll (1987).

Pseudo-likelihood and weighted squared residual estimation are based upon the method of moments. Nelder & Pregibon (1987) instead attempt to define a family of distributions with mean and variance functions given by (1.1), this class including as special cases skewed distributions such as the Poisson or gamma. Their extended quasi-likelihood is

$$\ell_{QL}(\beta, \theta, \sigma) = (-1/2) \sum_{i=1}^{N} [\log \{2\pi\sigma^2 g^2(y_i, z_i, \theta)\} + D\{y_i, \mu_i(\beta), z_i\}/\sigma^2], \quad (1.4)$$

where

$$D(y, \mu, \theta) = -2 \int_{y}^{\mu} \frac{y - w}{g^2(w, z, \theta)} \, dw.$$

The function ℓ_{QL} is sometimes but not always an exact log-likelihood. Under (1.2), if $\theta = 0$, ℓ_{QL} is the normal log-likelihood; for $\theta = 1.5$, that of the inverse Gaussian. For $\theta = .5$, $\sigma = 1$, ℓ_{QL} differs from the Poisson log-likelihood by replacing $y_i!$ by its Stirling approximation; for $\theta = 1$, ℓ_{QL} differs from the gamma log-likelihood by a factor depending on σ. One motivation for (1.4) is the Edgeworth expansion of Barndorff-Nielsen and Cox (1979) or the related saddlepoint approximation of Daniels (1954), which yield an expansion for the density of the mean of m random variables from a one parameter exponential family as $m \to \infty$. The leading term of the expansion at $m = 1$ is the extended quasi-likelihood summand. See Efron (1986) for a related formulation. Note that the form of ℓ_{QL} may be unsatisfactory in situations for which $g(y,z,\theta) = 0$ for $y = 0$. In this case Nelder & Pregibon suggest replacing $g(y,z,\theta)$ by $g(y+c,z,\theta)$ for some c; we use this adjustment where applicable in our discussion.

An additional reason for considering approximate likelihoods for a mean-variance model is that linear exponential families with given mean-variance relationship do not always exist. For example, Bar-Lev & Enis (1986) have shown that if the distribution of y_i is an exponential family with variance function (1.2), it is necessary that $\theta \notin (-\infty,0) \cup (0,1/2)$, so that such a family exists only when $\theta \in \{0\} \cup [1/2,\infty)$, and the general form for the density parameterized in terms of θ and σ is unwieldy.

We have observed in many examples that the pseudo-likelihood and quasi-likelihood methods lead to similar estimates, although sometimes inferences for θ are substantially different. In Section 2, we construct an asymptotic theory for extended quasi-likelihood which allows an easy illustration of the relationship between the two methods and suggests a simple motivation for the form of extended quasi-likelihood. We show that

under certain conditions the two estimators are asymptotically equivalent, although extended quasi-likelihood can be affected by an asymptotic bias while pseudo-likelihood is not when the underlying distribution is asymmetric. In Section 3 we discuss inference for θ based on the two approaches. From the theory of Section 2 we observe that while inference based on asymptotic theory for the two approaches may yield similar results under some conditions, such a test based on extended quasi-likelihood can be adversely affected by possible asymptotic bias of the estimator, a problem not shared by pseudo-likelihood tests.

2. SOME ASYMPTOTIC RESULTS

Neither pseudo-likelihood nor extended quasi-likelihood are exact likelihood approaches. Pseudo-likelihood is based on the method of moments, so that the estimating equations are unbiased and hence consistency and asymptotic normality obtain under very general conditions even without the assumption of normality. Let $v(\mu_i, z_i, \theta) = \log g(\mu_i, z_i, \theta)$, $v_\theta(\mu_i, z_i, \theta)$ be its column vector of partial derivatives with respect to θ, $\omega_\theta(\mu_i, z_i, \theta) = v_\theta(\mu_i, z_i, \theta) - N^{-1}\Sigma\, v_\theta(\mu_j, z_j, \theta)$, and $\xi(\mu, \theta)$ be the limiting covariance matrix of the v_θ. Let subscripts denote differentiation with respect to the argument, e.g., $g_\mu(\mu_i, z_i, \theta) = \partial g(\mu_i, z_i, \theta)/\partial \mu_i$. Define the errors $\epsilon_i = (y_i - \mu_i)/\{\sigma\, g(\mu_i, z_i, \theta)\}$, and assume the $\{\epsilon_i\}$ are independent with skewness ζ_i and kurtosis κ_i; $\kappa_i = 0$ for normality. Let $\tau = (\eta, \theta^t)^t$ and use subscripts PL and QL to denote pseudo-likelihood and extended quasi-likelihood, respectively.

RESULT 1 (Davidian and Carroll, 1987). Suppose that $(\hat{\beta}_* - \beta)/\sigma = O_p(N^{-1/2})$ and $\hat{\gamma}_{PL} - \gamma = O_p(N^{-1/2})$. Then $\hat{\theta}_{PL}$ is asymptotically normally distributed with mean θ. If $\sigma \to 0$ simultaneously with $N \to \infty$, then

$$N^{1/2}(\hat{\theta}_{PL} - \theta) = \{2\xi(\mu,\theta)\}^{-1} N^{-1/2} \sum_{i=1}^{N} (\epsilon_i^2 - 1)\omega_\theta(\mu_i, z_i, \theta) + o_p(1). \tag{2.1}$$

If the $\{\epsilon_i\}$ are identically distributed with kurtosis κ, the covariance matrix of the asymptotic distribution of $\hat{\theta}_{PL}$ is given by

$$(2 + \kappa)\ \{4N\ \xi(\mu,\theta)\}^{-1}. \tag{2.2}$$

The assumption $\sigma \to 0$ is a useful simplification technically and is relevant in applications where σ is "small" relative to the means as in radioimmunoassay, see Carroll, Davidian & Smith (unpublished). In the gamma and lognormal distributions, σ is the coefficient of variation, which is often fairly small. Alternatively, think of y_i as the mean of m observations with mean μ_i and variance $g^2(\mu_i, z_i, \theta)$, equate σ and $m^{-1/2}$, and let $m \to \infty$.

The assumption $\sigma \to 0$ yields a motivation for (1.4). Since the goal of extended quasi-likelihood is to describe a class of distributions "nearly" containing exponential families, consider a density h such that

$$\log h(y,\alpha,\theta,\sigma) = \{y\alpha - b(\alpha)\}/\sigma^2 + c(y,\theta\ \sigma) \tag{2.3}$$

for some b, c, and $\alpha = \alpha(\mu,\theta)$. To satisfy (1.1) we require $\partial b(\alpha)/\partial\alpha = \mu$

and $\partial^2 b(\alpha)/\partial\alpha^2 = g^2(\mu,z,\theta)$, implying that $\mu = \{\partial b(\alpha)/\partial\mu\}\ \partial\mu/\partial\alpha$ and $g^2(\mu,z,\theta) = \partial\mu/\partial\alpha$. This yields, writing b now as a function of μ,

$$\alpha = \int_{-\infty}^{\mu} \{1/g^2(u,z,\theta)\}\ du; \qquad b(\mu) = \int_{-\infty}^{\mu} \{u/g^2(u,z,\theta)\}\ du.$$

Plugging into (2.3) gives after simplification

$$\log h(y,\alpha,\sigma) = -\ (2\sigma^2)^{-1}\ D(y,z,\theta) + d(y,\theta,\sigma) \qquad (2.4)$$

for some function d. For h to be a density we must choose d so that h integrates to one; by approximating the first term on the right side of (2.4) when σ is small we may approximate d. Since when σ is small we have $\sigma^{-2}\ D(y,\mu,z) \approx (y-u)^2/\{\sigma^2 g^2(y,z,\theta)\}$, $d \cong -(1/2)\ \log\{2\pi\sigma^2 g^2(y,z,\theta)$. Inserting this in (2.4) yields the summand of (1.4).

The fact that (1.4) is an approximate log-likelihood implies that $\hat{\theta}_{QL}$ need not be consistent. With the suggested adjustment c = 1/6 as in Nelder & Pregibon (1987), if the $\{y_i\}$ are dist. ted as Poisson with means $\{\mu_i\}$ taking on values 1 and 4 in equal proportions, the theory of M-estimation as in Huber (1981 p. 130-132) implies that $\hat{\theta}_{QL}$ converges to 0.675; if the $\{\mu_i\}$ take values 1 and 5, $\hat{\theta}_{QL}$ converges to 0.640. If the $\{\mu_i\}$ take on larger values, such as 30, 40 and 50 in equal proportions, however, $\hat{\theta}_{QL}$ converges to 0.500.

Since the estimating equation for the extended quasi-likelihood estimate $\hat{\theta}_{QL}$ can be biased, standard asymptotic theory for $\hat{\theta}_{QL}$, while possible to construct, is not fully informative. As an approximation we use the small σ assumption to construct an asymptotic theory. We also describe an approach suggested by the Poisson case for "large" $\{\mu_i\}$.

RESULT 2. Suppose that $N^{1/2}(\hat{\tau}_{QL} - \tau) = O_p(1)$ and $N^{1/2}(\hat{\beta}_{QL} - \beta)/\sigma = O_p(1)$ if $N^{1/2}\sigma \to \lambda \geq 0$ as $N \to \infty$, $\sigma \to 0$. Then

$$N^{1/2}(\hat{\theta}_{QL} - \theta) = \{2\,\xi(\mu,\theta)\}^{-1} N^{-1/2} \sum_{i=1}^{N} (\epsilon_i^2 - 1)\,\omega_\theta(\mu_i, z_i, \theta)$$
$$+ (N^{1/2}\sigma)\,\{6\,\xi(\mu,\theta)\}^{-1} C_N + o_p(1), \text{ where} \qquad (2.5)$$
$$C_N = N^{-1} \sum_{i=1}^{N} \zeta_i \{g(\mu_i, z_i, \theta)\, v_{\theta\mu}(\mu_i, z_i, \theta) - 2\, g_\mu(\mu_i, z_i, \theta)\, \omega_\theta(\mu_i, z_i, \theta)\}.$$

A sketch of the proof is contained in the Appendix. The implication of (2.5) is that while $\hat{\theta}_{PL}$ and $\hat{\theta}_{QL}$ behave similarly, they differ in an asymptotic fashion through the second term on the right hand side of (2.5) in a way that might affect asymptotic inference. For example, if the $\{\epsilon_i\}$ are identically distributed with skewness ζ and kurtosis κ, then $\hat{\theta}_{QL}$ is asymptotically normal with covariance (2.2) and mean

$$\theta + \{6N^{1/2}\xi(\mu,\theta)\}^{-1}\{\lambda\zeta C\}, \text{ where } C_N \to C.$$

From (2.5), $\hat{\theta}_{QL}$ and $\hat{\theta}_{PL}$ will be asymptotically equivalent only if $\lambda = 0$ or $C_N \to 0$; the latter will occur for symmetrically distributed data.

In the case of (1.2), $v_\theta(\mu_i, z_i, \theta) = \log \mu_i$ so that

$$C_N = N^{-1} \sum_{i=1}^{N} \zeta_i\, \mu_i^{\theta-1}\{1 - 2\theta\,(\log \mu_i - \bar{l}_N)\}, \quad \bar{l}_N = \sum_{j=1}^{N} \log \mu_j.$$

For the normal distribution, $\zeta_i = \kappa_i = 0$; for the gamma, lognormal, and inverse Gaussian distributions $\zeta_i = O(\sigma)$ and $\kappa_i = O(\sigma^2)$, so that the asymptotic bias is 0 and the two estimators are asymptotically equivalent with covariance the same as if the data were normally distributed with mean

μ_i and variance $\sigma^2\mu_i^{2\theta}$. From Bar-Lev & Enis (1986), $\zeta_i = O(\sigma)$ for distributions which are exponential families with $\theta \in \{0\} \cup [1/2,\infty)$. If the $\{y_i\}$ are not from an exponential family, the asymptotic bias need not be zero. For example, consider a shifted gamma model $y_i = \mu_i + \sigma g(\mu_i,z_i,\theta)\epsilon_i$, where w_i has a gamma (α_i,φ_i) distribution with $E(w_i) = \alpha_i/\varphi_i$, and $\epsilon_i = \{w_i - (\alpha_i/\varphi_i)\}\ (\alpha_i/\varphi_i^2)^{-1/2}$, so that $E(\epsilon_i^2) = 1$. In this case $\zeta_i = 2\ \alpha_i^{-1/2}$, so that if the $\{\alpha_i\}$ do not depend on σ, the asymptotic bias will not vanish.

An asymptotic theory for which the means are "large" in which σ remains fixed yields a similar result under (1.2) if $\theta < 1$. Let $\mu_{0,N}$ be a sequence to be chosen shortly. Define $\mu_i^* = \mu_i/\mu_{0,N}$ and $y_i^* = y_i/\mu_{0,N}$ so that $\epsilon_i = \{y_i^* - \mu_i^*\}/(\delta\ \mu_i^{*\theta})$, where $\delta = \sigma\ \mu_{0,N}^{(\theta-1)}$. If as $N \to \infty$, $\min_i \mu_i \to \infty$ and $\mu_{0,N} \to \infty$ in such a way that the $\{\mu_i^*\}$ and the $\{y_i^*\}$ are well-behaved, then if $\theta < 1$, $\delta \to 0$ as $N \to \infty$ so that the calculations here parallel those for the case of small σ. By analogy, the small σ part of Result 1 holds. Replacing σ by δ in (2.5), in the Poisson case for which $\theta = .5$ and $\sigma = 1$, $\zeta_i = \mu_i^{-1/2}$ and $\kappa_i = \mu_i^{-1}$, so that $C_N \to 0$ and the limiting covariance of $\hat{\theta}_{QL}$ is as if $\kappa_i = 0$. Thus, in the case of "large" means and data distributed as Poisson, extended quasi-likelihood and pseudo-likelihood will behave similarly.

3. INFERENCE FOR θ

The asymptotic distribution theory of Section 2 can be used to construct tests of H_0: $\theta = \theta_0$. For simplicity consider identically distributed $\{\epsilon_i\}$ with kurtosis κ. From (1.3), $\hat{\theta}_{PL}$ maximizes $\ell_{PL}^*(\hat{\beta}_*,\theta)$, where

$$\ell^*_{PL}(\beta,\theta) = -N \log \hat{\sigma}_{PL}(\beta,\theta) - \sum_{i=1}^{N} \log g\{\mu_i(\beta), z_i, \theta\},$$

$$\hat{\sigma}^2_{PL}(\beta,\theta) = \sum_{i=1}^{N} \{(y_i - f(x_i,\beta)\}^2 / g^2\{\mu_i(\beta), z_i, \theta\}.$$

One might reasonably base inference for θ on a test statistic

$$T_N = -2\,[\ell^*_{PL}\{\hat{\beta}(\theta_0), \theta_0\} - \ell^*_{PL}\{\hat{\beta}(\hat{\theta}_{PL}), \hat{\theta}_{PL}\}],$$

where $\hat{\beta}(\theta)$ denotes a generalized least squares estimate computed at θ. Result 1 may be used to show that under H_0, $\{2/(2+\kappa)\}\ T_N$ is asymptotically distributed as χ^2_r, so that a test based on this statistic with κ appropriately estimated is an asymptotic α-level test. McCullagh & Pregibon (1987) consider estimators for the cumulants for linear regression models.

Nelder & Pregibon (1987) suggest a likelihood ratio type test based on treating the extended quasi-likelihood as an actual likelihood. Such a test is based on

$$Q_N = -2\,[\ell^*_{QL}\{\hat{\beta}(\theta_0), \theta_0\} - \ell^*_{QL}\{\hat{\beta}(\hat{\theta}_{QL}), \hat{\theta}_{QL}\}],$$

where

$$\ell^*_{QL}(\beta,\theta) = -N \log \sigma_{QL}(\beta,\theta) - \sum_{i=1}^{N} \log g(y_i, z_i, \theta),$$

$$\sigma^2_{QL}(\beta,\theta) = N^{-1} \sum_{i=1}^{N} D\{y_i, \mu_i(\beta), z_i\}.$$

In the situation of Result 2, under H_0 $\{2/(2+\kappa)\}\ Q_N$ is asymptotically distributed as noncentral χ^2_r with noncentrality parameter $\Delta = \lambda^2\zeta^2 C^t \xi(\mu,\theta)^{-1} C\ \{9(2+\kappa)\}^{-1}$. As long as $\Delta = 0$, comparing this statistic to

the percentiles of the χ^2_r distribution is an asymptotic α-level test which is asymptotically equivalent to the test based on T_N. Nelder & Pregibon suggest comparing Q_N directly to the percentiles of the χ^2_r distribution, not accounting for the factor $2/(2+\kappa)$. In the saddlepoint approximation approach, $m \to \infty$ implies $\kappa \to 0$, thus they observe that if the underlying distribution of the data is known to be from an exponential family, then such a test is asymptotically valid. In our asymptotics, for the cases of the normal, Poisson, gamma, and inverse Gaussian examples cited in Section 2 we see this to be the case. We further obtain the correct form and properties for a test of this type when only the mean-variance relationship is specified.

For a model such as (1.1) for which only the mean and variance are specified, interest in θ may be in the context of trying to understand the structure of the variances, not the form of the underlying distribution. A test based on Q_N will approach its nominal level only if $\Delta = 0$. Thus, when the underlying distribution of the data is such that $\hat{\theta}_{QL}$ is biased asymptotically so that $\Delta \neq 0$, in the case of a nonsymmetric error distribution for example, the validity of a χ^2 test based on Q_N may be seriously affected in that there will be bias in the asymptotic levels of the test.

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APPENDIX : SKETCH OF PROOF OF RESULT 2

For convenience, let $\eta = \log \sigma$ and let $(\hat{\beta},\hat{\theta},\hat{\eta})$ be the joint extended quasi-likelihood estimators for (β,θ,η). Let $\tau(i,\beta,\theta) = \{1 \ , \ v_\theta^t(\mu_i,z_i,\theta)\}^t$, $\rho(i,\beta,\theta) = f_\beta(x_i,\beta)/g\{\mu_i(\beta),z_i,\theta\}$, $B_N = -4N^{-1}\Sigma\ \tau(i,\beta,\theta)\ \tau(i,\beta,\theta)^t$, $J_N = 2N^{-1}\Sigma\ \epsilon_i\ \rho(i,\beta,\theta)\ \tau(i,\beta,\theta)^t$ and $G_N = N^{-1}\Sigma\ \rho(i,\beta,\theta\ \rho(i,\beta,\theta)^t$. From (1.4), $(\hat{\beta},\hat{\theta},\hat{\eta})$ solves

$$0 = \begin{bmatrix} Q_{1,N}(\hat{\beta},\hat{\theta},\hat{\eta}) \\ Q_{2,N}(\hat{\beta},\hat{\theta},\hat{\eta}) \\ Q_{3,N}(\hat{\beta},\hat{\theta},\hat{\eta}) \end{bmatrix}, \quad \text{where} \tag{A.1}$$

$$Q_{1,N}(\beta,\theta,\eta) = N^{-1/2} \sum_{i=1}^{N} e^{-2\eta}\ (Y_i - \mu_i)\ \rho(i,\beta,\theta)/g(\mu_i,z_i,\theta);$$

$$Q_{2,N}(\beta,\theta,\eta) = N^{-1/2} \sum_{i=1}^{N} \{\ e^{-2\eta}\ D(Y_i,\mu_i,z_i) - 1\};$$

$$Q_{3,N}(\beta,\theta,\eta) = N^{-1/2} \sum_{i=1}^{N} [\ -\frac{1}{2}\ e^{-2\eta}\ \partial D(Y_i,\mu_i,z_i)/\partial\theta - \{\partial \log\ g(Y_i,z_i,\theta)/\partial\theta\}].$$

The following result is shown by assuming appropriate smoothness conditions for g so that D may be differentiated.

LEMMA A. Under regularity conditions,

$$N^{-1} \sum_{i=1}^{N} D(Y_i,\mu_i,z_i) = \sigma^2\ N^{-1} \sum_{i=1}^{N} \epsilon_i^2 + \sigma^3\ N^{-1} \sum_{i=1}^{N} s_{1,i}\ \epsilon_i^3 + O_p(\sigma^4);$$

$$N^{-1} \sum_{i=1}^{N} \partial D(Y_i,\mu_i,z_i)/\partial\theta = -2 \{ \sigma^2 N^{-1} \sum_{i=1}^{N} \epsilon_i^2 \upsilon_\theta(i,\beta,\theta) + \sigma^3 N^{-1} \sum_{i=1}^{N} s_{2,i} \epsilon_i^3 \} + O_p(\sigma^4);$$

$$N^{-1} \sum_{i=1}^{N} \partial^2 D(Y_i,\mu_i,z_i)/\partial\theta\partial\theta^t = 2 \sigma^2 N^{-1} \sum_{i=1}^{N} \epsilon_i^2 [3 \upsilon_\theta(i,\beta,\theta) \upsilon_\theta^t(i,\beta,\theta) - \{g_{\theta\theta}(\mu_i,z_i,\theta) / g(\mu_i,z_i,\theta)\}] + O_p(\sigma^3),$$

where $g_{\theta\theta}(\mu_i,z_i,\theta) = \partial^2\{g(\mu_i,z_i,\theta)\}/\partial\theta\partial\theta^t$, $s_{1,i} = -2\, g_\mu(\mu_i,z_i,\theta)/3$, and $s_{2,1} = g_{\theta\mu}(\mu_i,z_i,\theta)/3 - \upsilon_\theta(\mu_i,z_i,\theta)\, g_\mu(\mu_i,z_i,\theta)$.

A Taylor series in (A.1) using consistency, a Taylor series in σ about 0 using Lemma A and laws of large numbers yield after simplification

$$\begin{bmatrix} C_N & J_N \\ J_N & \frac{1}{2} B_N \end{bmatrix} N^{1/2} \begin{bmatrix} (\hat{\beta} - \beta)/\sigma \\ \hat{\eta} - \eta \\ \hat{\theta} - \theta \end{bmatrix} \tag{A.2}$$

$$= \begin{bmatrix} N^{-1/2} \sum_{i=1}^{N} \epsilon_i\, \rho(1,\beta,\theta) \\ N^{-1/2} \sum_{i=1}^{N} (\epsilon_i^2 - 1)\, \tau(i,\beta,\theta) \end{bmatrix} + (N^{1/2}\sigma) \begin{bmatrix} 0 \\ N^{-1} \sum_{i=1}^{N} \epsilon_i^3 s_i \end{bmatrix} + o_p(1),$$

where $s_i^t = (s_{1,i}, s_{2,i}^t)$. Equation (A.2) implies that, as $N \to \infty$, $\sigma \to 0$,

$$B_N N^{1/2}\begin{bmatrix} \hat{\eta} - \eta \\ \hat{\theta} - \theta \end{bmatrix} = 2 N^{-1/2} \sum_{i=1}^{N} (\epsilon_i^2 - 1) \tau(i,\beta,\theta)$$

$$+ 2 (N^{1/2}\sigma) N^{-1} \sum_{i=1}^{N} \epsilon_i^3 s_i + o_p(1).$$

Algebra and simple probability limit calculations yield the result. Equation (A.2) also shows that in these asymptotics $\hat{\beta}_{QL}$ is equivalent to a generalized least squares estimator for β. □

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